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Robert M. Kirby, Robert Haimes

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University of Utah
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14. ABSTRACT
High-order finite element methods (also known as spectral/hp element methods) using either the continuous Galerkin or discontinuous Galerkin formulation have reached a level of sophistication such that they are now commonly applied to a diverse set of real-life engineering problems. Visualization of computed results is often used as a means of understanding and evaluating the numerical approximation of the mathematical model, and it provides a means of "closing the loop" – that is, of critically evaluating the computational results for refinement of the model and/or numerics or for interpretation of the physical world. Visualizations of high-order finite element results which do not respect the a priori knowledge of how the data were

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a. REPORT	b. ABSTRACT	c. THIS PAGE			Robert Kirby
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Report Title

VISUALIZATION OF HIGH-ORDER FINITE ELEMENT METHODS

ABSTRACT

High-order finite element methods (also known as spectral/hp element methods) using either the continuous Galerkin or discontinuous Galerkin formulation have reached a level of sophistication such that they are now commonly applied to a diverse set of real-life engineering problems. Visualization of computed results is often used as a means of understanding and evaluating the numerical approximation of the mathematical model, and it provides a means of "closing the loop" – that is, of critically evaluating the computational results for refinement of the model and/or numerics or for interpretation of the physical world. Visualizations of high-order finite element results which do not respect the a priori knowledge of how the data were produced and which do not provide a quantification of the visual error produced undermine the scientific process just described. The goals of this effort are to define, investigate, and address the technical obstacles inherent in visualization of data derived from high-order numerical methods and to develop algorithms and software solutions that can be employed by the high-order simulation community.

List of papers submitted or published that acknowledge ARO support during this reporting period. List the papers, including journal references, in the following categories:

(a) Papers published in peer-reviewed journals (N/A for none)

1) Miriah Meyer, Blake Nelson, Robert M. Kirby and Ross Whitaker, "Particle Systems for Efficient and Accurate Finite Element Visualization", IEEE Transactions on Visualization and Computer Graphics, Vol. 13, Number 5, pages 1015-1026, 2007.

2) Miriah Meyer, Robert M. Kirby and Ross Whitaker, "Topology, Accuracy, and Quality of Isosurface Meshes Using Dynamic Particles", IEEE Transactions on Visualization and Computer Graphics (IEEE Visualization Issue), Vol. 13, Number 6, pages 1704-1711, 2007.

3) Sean Curtis, Robert M. Kirby, Jennifer K. Ryan and Chi-Wang Shu, "Post-processing for the Discontinuous Galerkin Method Over Non-Uniform Meshes", SIAM Journal of Scientific Computing, Vol. 30, Number 1, pages 272-289, 2007.

4) Michael Steffen, Sean Curtis, Robert M. Kirby and Jennifer K. Ryan, "Investigation of Smoothness-Increasing Accuracy-Conserving Filters for Improving Streamline Integration Through Discontinuous Fields", IEEE Transactions on Visualization and Computer Graphics, Vol. 14, Number 3, pages 680-692, 2008.

5) Miriah Meyer, Ross Whitaker, Robert M. Kirby, Christian Ledergerber and Hanspeter Pfister, "Particle-based Sampling and Meshing of Surfaces in Multimaterial Volumes", IEEE Transactions on Visualization and Computer Graphics (IEEE Visualization Issue), Accepted for Publication, 2008.

6) David Walfisch, Jennifer K. Ryan, Robert M. Kirby and Robert Haimes, "One-Sided Smoothness-Increasing Accuracy-Conserving Filtering for Enhanced Streamline Integration through Discontinuous Fields", Journal of Scientific Computing, Accepted for Publication, 2008.

Number of Papers published in peer-reviewed journals: 6.00

(b) Papers published in non-peer-reviewed journals or in conference proceedings (N/A for none)

Number of Papers published in non peer-reviewed journals: 0.00

(c) Presentations

1) Intelligent Visualization and Simulation Lab, University of Kaiserslautern, Germany. Presented a talk entitled “Visualization of High-Order Finite Element Methods", June 2008.

2) Center of Complex Systems and Visualization, University of Bremen, Germany. Presented a talk entitled “Topology, Accuracy, and Quality of Isosurface Meshes Using Dynamic Particles", February 2008.

3) International Workshop on High-Order Finite Element Methods, Herrsching am Ammersee (near Munich), Germany. Presented a talk entitled ``Visualization of High Order Finite Element Methods", May 2007.

4) Center of Complex Systems and Visualization, University of Bremen, Germany. Presented a talk entitled ``Particle Systems for Efficient and Accurate High-Order Finite Element Visualization", March 2007.

5) Sean Curtis, Robert M. Kirby and Jennifer K. Ryan, “Accuracy Enhancing Filtering With Application To Visualization”. Presented at the 7th World Congress on Computational Mechanics, July 2006.

6) Miriah Meyer, Blake Nelson, Robert M. Kirby and Ross Whitaker, ``Particle Systems for Efficient and Accurate High-Order Finite Element Visualization", International Conference on Spectral and High-Order Methods, June 2007.

Number of Presentations: 6.00

Non Peer-Reviewed Conference Proceeding publications (other than abstracts):

Number of Non Peer-Reviewed Conference Proceeding publications (other than abstracts): 0

Peer-Reviewed Conference Proceeding publications (other than abstracts):

Number of Peer-Reviewed Conference Proceeding publications (other than abstracts): 0

(d) Manuscripts

Number of Manuscripts: 0.00

Number of Inventions:

Graduate Students

NAME	PERCENT_SUPPORTED
Sarah Geneser	1.00
Miriah Meyer	1.00
David Walfisch	0.50
FTE Equivalent:	**2.50**
Total Number:	**3**

Names of Post Doctorates

NAME	PERCENT_SUPPORTED
FTE Equivalent:	
Total Number:	

Names of Faculty Supported

NAME	PERCENT_SUPPORTED	National Academy Member
Mike Kirby	0.43	No
Robert Haimes	0.93	No
FTE Equivalent:	**1.36**	
Total Number:	**2**	

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Names of Personnel receiving masters degrees

NAME	
David Walfisch	
Total Number:	**1**

Names of personnel receiving PHDs

NAME	
Sarah Geneser	
Miriah Meyer	
Total Number:	**2**

Names of other research staff

NAME	PERCENT_SUPPORTED
FTE Equivalent:	
Total Number:	

Sub Contractors (DD882)

1 a. Massachusetts Institute of Technology

1 b. 77 Massachusetts Ave.
Building 32-D670
Cambridge MA 021394307

Sub Contractor Numbers (c): 2411099

Patent Clause Number (d-1):

Patent Date (d-2):

Work Description (e):

Sub Contract Award Date (f-1): 8/1/2005 12:00:00AM

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Sub Contract Award Date (f-1): 8/1/2005 12:00:00AM

Sub Contract Est Completion Date(f-2): 7/31/2008 12:00:00AM

Inventions (DD882)

VISUALIZATION OF HIGH-ORDER FINITE ELEMENT METHODS

ARO W911NF-05-1-0395, Dr. Mike Coyle

Robert M. Kirby
School of Computing
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Robert Haimes
Department of Aeronautics & Astronautics
Massachusetts Institute of Technology

Abstract: High-order finite element methods (also known as spectral/hp element methods) using either the continuous Galerkin or discontinuous Galerkin formulation have reached a level of sophistication such that they are now commonly applied to a diverse set of real-life engineering problems. Visualization of computed results is often used as a means of understanding and evaluating the numerical approximation of the mathematical model, and it provides a means of "closing the loop" – that is, of critically evaluating the computational results for refinement of the model and/or numerics or for interpretation of the physical world. Visualizations of high-order finite element results which do not respect the a priori knowledge of how the data were produced and which do not provide a quantification of the visual error produced undermine the scientific process just described. The goals of this effort are to define, investigate, and address the technical obstacles inherent in visualization of data derived from high-order numerical methods and to develop algorithms and software solutions that can be employed by the high-order simulation community.

Statement of Problem Studies

The goals of this effort are to define, investigate, and address the technical obstacles inherent in visualization of data derived from high-order numerical methods and to develop algorithms and software solutions that can be employed by the high-order simulation community.

Summary of Results

In this section, we first present the motivating work (published in [R1] by one of the investigators) for our previous ARO grant and then provide a summary of results as a consequence of ARO funding. To summarize – six peer-reviewed journal articles have been published or accepted for publication: four articles targeting the visualization community [J1, J2, J4, J5] and two targeting the computational mathematics community [J3, J6].

• Isosurface Visualization

Our initial work on high-order finite element visualization was motivated by the work of Nelson and Kirby [R1], in which they presented an algorithm for ray-casting high-order, spectral/*hp* elements. Their method uses a world-space approximation of the composition of the coordinate transformation and the reference space basis functions. It assumes multi-linear mappings (linear element boundaries in world space), and includes a quantification of the approximation and root-finding error. They show that the image-space method compares favorably with marching cubes in compute time when the tolerances on surface position are sufficiently high. Figure 1 provides an example of the type of visualizations produced by their work. The marching cubes image (left) was generated by sampling the finite element volume on a rectilinear grid of spacing h, using a marching cubes algorithm to provide a tessellated isosurface, and rendering the triangular isosurface using ray-casting (since the marching cubes result is a triangular mesh, the ray-casting can be done exactly as done in [R2]). For the marching cubes image presented, a grid spacing of h=0.015 (yielding 4,705,274 voxels) was used. For the high-order ray-traced image (right), mapping inversion error of 10^{-8} and 11^{th} order projected polynomials were used. These parameters were chosen such that the spectral/*hp* element evaluation time and rendering time was nearly identical to generate the two images. The root-mean-square error for the marching cubes image is 0.0158; the root-mean-square error for the ray-traced image is 3.5e-11. The images look very similar, however the root-mean-square error difference between the images is significant. We should also point out that the file size for the marching cubes representation is over an order of magnitude larger than the high-order representation.

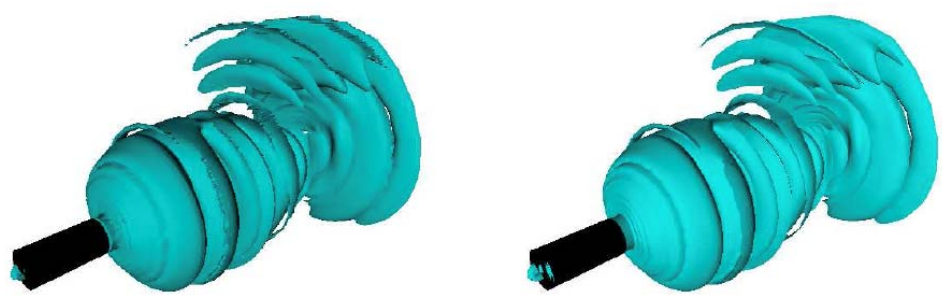

Figure 1: Marching cubes image with h=0.015 corresponding to 4,705,274 voxels (left) and ray-traced solution using 11^{th} order projected polynomials (right) for isosurface of pressure at C = 0.0 chosen such that the spectral/*hp* element data evaluation and rendering time is nearly identical (on the order of 200 seconds). The root-mean-square error for the marching cubes image is 0.0158; the root-mean-square error for the ray-traced image is 3.5e-11.

Although the ray-casting methodology provided a “pixel exact” visualization of the isosurface, it did so in what is referred to as “image space”. This implies that even after a researcher found the isosurface of interest which they wanted to examine, each rotation, translation or zoom into the image required approximately the same amount of rendering time as each pixel's color has to be recomputed.

The classic way to attempt to solve this issue is to render things in “object space” – that is, to generate objects (triangles, for instance) on the isosurface so that once an isosurface is found and an object is created, its rendering can be done quickly. In [J1] we proposed visualizing isosurfaces in high-order finite element datasets with a particle system as a means of solving this problem. We presented a framework that allows particles to sample an isosurface in reference space, avoiding the costly inverse mapping of positions from world space when evaluating the basis functions. The distribution of particles across the reference space isosurface is controlled by geometric information from the world space isosurface, such as the surface gradient and curvature. The resulting particle distributions can be distributed evenly or adapted to accommodate world-space surface features. This provides compact, efficient, and accurate isosurface representations of these challenging data sets. In Figure 2 we present a visualization of an isosurface of pressure within an incompressible fluid flow field.

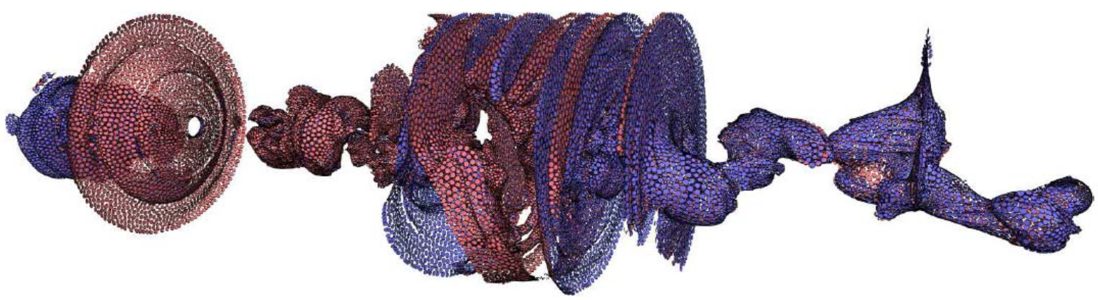

Figure 2: An isosurface of a finite element fluid simulation pressure field sampled with a particle system. The color indicates the relative direction of the surface normal at the particle (blue indicates *outward* and red indicates *inward*). }

When one employs objects to mark or denote an isosurface, one faces the challenge of knowing how many objects to use and how densely to pack them. A sparse packing of the objects can miss critical features of the isosurface. A dense packing can be very inefficient (especially when the density is much higher than is needed). In [J2], we describe a method for constructing isosurface triangulations of sampled, volumetric, three-dimensional scalar fields that attempts to tackle this sampling density problem. The resulting meshes consist of triangles that are of consistently high quality, making them well suited for accurate interpolation of scalar and vector-valued quantities, as required for numerous applications in visualization and numerical simulation. The proposed method does not rely on a local construction or adjustment of triangles as is done, for instance, in advancing wavefront or adaptive refinement methods. Instead, a system of dynamic particles optimally samples an implicit function such that the particles' relative positions can produce a topologically correct Delaunay triangulation. Thus, the proposed method relies on a *global* placement of triangle vertices. The main contributions of this work was the integration of dynamic particles systems with surface sampling theory and PDE-based methods for controlling the local variability of particle densities, as well as detailing a practical method that accommodates Delaunay sampling requirements to generate sparse sets of points for the production of high-quality tessellations. In [J5] we extended this work to handle surfaces that come as a consequence of multi-material interfaces.

• Streamline Integration

A quick search of both the visualization and the application domain literature demonstrates that streamlines are a popular visualization tool, second only to isosurfaces. The bias toward using streamlines is in part explained by studies that show streamlines to be effective visual representations for elucidating the salient features of the vector fields [R3]. Furthermore, streamlines as a visual representation are appealing because they are applicable for both two-dimensional and three-dimensional fields [R4]. It was for this reason that we invested time considering how streamlining would be impacted by high-order finite element data.

Streamline integration is often accomplished through the application of ordinary differential equation (ODE) integrators such as predictor-corrector or Runge-Kutta schemes. The foundation for the development of these schemes is the use of Taylor series for building numerical approximations of the solution of the ODE of interest.
Taylor series can be further used to elucidate the error characteristics of the derived scheme. All schemes employed for streamline integration that are built using such an approach exhibit error characteristics which are predicated on the smoothness of the field through which the streamline is being integrated.

Low-order and high-order finite volume and finite element fields are among the most common types of fluid flow simulation datasets available. Streamlining is commonly applied to these datasets. The property of these fields which challenges classic streamline integration using Taylor series based approximations is that finite volume fields are piecewise discontinuous and finite element fields are only C^0 continuous. Hence one of the limiting factors of streamline accuracy and integration efficiency is the lack of smoothness at the inter-element level of finite volume and finite element data.

Adaptive error control techniques are often used to ameliorate the challenge posed by inter-element discontinuities. To paraphrase a classic work on the subject of solving ODEs with discontinuities [R5], one must (1) detect, (2) determine the order, size and location of, and (3) judiciously “pass over” discontinuities for effective error control. Such an approach has been effectively employed within the visualization community for overcoming the challenges posed by discontinuous data at the cost of increased number of evaluations of the field data. The number of evaluations of the field increases drastically with every discontinuity that is encountered [R5]. Thus if one requires a particular error tolerance and employs such methods for error control when integrating a streamline through a finite volume or finite element dataset, a large amount of the computational work involved is due to handling inter-element discontinuities and not the intra-element integration. We demonstrate this in Figure 3 where one can see that the number of streamline sampling steps goes up drastically each time a streamline attempts to traverse over an element boundary.

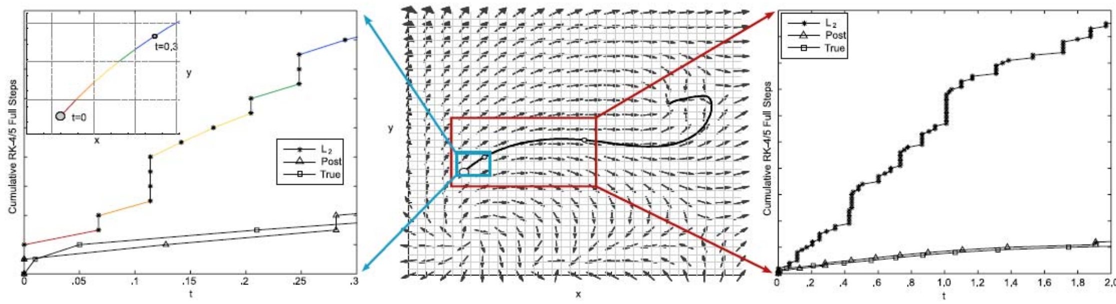

Figure 3: The center graph shows a streamline on an L_2 projected field integrated using RK-4/5. The left graph shows the streamline between t=0 and t=0.3 and the cumulative number of RK-4/5 steps (including rejects) required for integration. Vertical lines on this graph represent multiple rejected steps occurring when the streamline crosses element boundaries. The right graph shows the cumulative number of RK-4/5 steps required for integration to t=2.0.

As the root of the difficulties is the discontinuous nature of the data, one could speculate that if one were to filter the data in such a way that it was no longer discontinuous, streamline integration could then be made more efficient. The caveat that arises when one is interested in simulation and visualization error control is how does one select a filter that does not destroy the formal accuracy of the simulation data through which the streamlines are to be integrated? Recent mathematical advances [R6, R7] have shown that such filters can be constructed for high-order finite element and discontinuous Galerkin (high-order finite volume) data on uniform quadrilateral and hexahedral meshes. These filters are such that they have the provable quality that they increase the level of smoothness of the field without destroying the accuracy in the case that the “true solution” that the simulation is approximating is smooth. In fact, in many cases, these filters can increase the accuracy of the solution.

As part of our work, we investigated the use of such filters applied to discontinuous data prior to streamline integration, and found that they can drastically improve the computational efficiency of the integration process. We currently have two published papers on this topic [J3, J4] (one presenting this work to the visualization community, and one paper presenting new computational mathematics work which came as a consequence of this study). We also have an accepted paper in which we have adapted this idea to be more computationally efficient [J6]. We proposed a new technique that uses a one-dimensional convolution kernel to introduce continuity between elements, and increase smoothness while not introducing additional error in the solution. Furthermore, this one-dimensional implementation is the same regardless of the dimension of the simulation data. This in turn will aid in accomplishing the goals of visualization of data over more complex geometries while still improving the smoothness of the field and not compromising the accuracy of the data.

Supported Talks and Publications

Journal Publications

[J1]: Miriah Meyer, Blake Nelson, Robert M. Kirby and Ross Whitaker, "Particle Systems for Efficient and Accurate Finite Element Visualization", *IEEE Transactions on Visualization and Computer Graphics*, Vol. 13, Number 5, pages 1015-1026, 2007.

[J2]: Miriah Meyer, Robert M. Kirby and Ross Whitaker, "Topology, Accuracy, and Quality of Isosurface Meshes Using Dynamic Particles", *IEEE Transactions on Visualization and Computer Graphics* (IEEE Visualization Issue), Vol. 13, Number 6, pages 1704-1711, 2007.

[J3]: Sean Curtis, Robert M. Kirby, Jennifer K. Ryan and Chi-Wang Shu, "Post-processing for the Discontinuous Galerkin Method Over Non-Uniform Meshes", *SIAM Journal of Scientific Computing*, Vol. 30, Number 1, pages 272-289, 2007.

[J4]: Michael Steffen, Sean Curtis, Robert M. Kirby and Jennifer K. Ryan, "Investigation of Smoothness-Increasing Accuracy-Conserving Filters for Improving Streamline Integration Through Discontinuous Fields", *IEEE Transactions on Visualization and Computer Graphics,* Vol. 14, Number 3, pages 680-692, 2008.

[J5]: Miriah Meyer, Ross Whitaker, Robert M. Kirby, Christian Ledergerber and Hanspeter Pfister, "Particle-based Sampling and Meshing of Surfaces in Multimaterial Volumes", *IEEE Transactions on Visualization and Computer Graphics* (IEEE Visualization Issue), Accepted for Publication, 2008.

[J6]: David Walfisch, Jennifer K. Ryan, Robert M. Kirby and Robert Haimes, "One-Sided Smoothness-Increasing Accuracy-Conserving Filtering for Enhanced Streamline Integration through Discontinuous Fields", *Journal of Scientific Computing*, Accepted for Publication, 2008.

Invited Talks

Intelligent Visualization and Simulation Lab, University of Kaiserslautern, Germany. Presented a talk entitled "Visualization of High-Order Finite Element Methods", June 2008.

Center of Complex Systems and Visualization, University of Bremen, Germany. Presented a talk entitled "Topology, Accuracy, and Quality of Isosurface Meshes Using Dynamic Particles", February 2008.

International Workshop on High-Order Finite Element Methods, Herrsching am Ammersee (near Munich), Germany. Presented a talk entitled ``Visualization of High Order Finite Element Methods'', May 2007.

Center of Complex Systems and Visualization, University of Bremen, Germany. Presented a talk entitled ``Particle Systems for Efficient and Accurate High-Order Finite Element Visualization'', March 2007.

<u>Talks</u>

Sean Curtis, Robert M. Kirby and Jennifer K. Ryan, "Accuracy Enhancing Filtering With Application To Visualization". Presented at the 7th World Congress on Computational Mechanics, July 2006.

Miriah Meyer, Blake Nelson, Robert M. Kirby and Ross Whitaker, ``Particle Systems for Efficient and Accurate High-Order Finite Element Visualization'', International Conference on Spectral and High-Order Methods, June 2007.

Supported Individuals

Professor Robert M. Kirby (PI) – Utah

Mr. Robert Haimes (MIT subcontract PI) – MIT

Sarah Geneser (Spring/Summer 2006, Summer 2008) – Utah
Completed a PhD in Computer Science, Spring 2008, University of Utah

Miriah Meyer (Fall 2006 – Spring 2008) – Utah
Completed a PhD in Computer Science, Spring 2008, University of Utah

David Walfisch (Fall 2005 – Spring 2008), MIT
Completed a MS in Aeronautics, Spring 2008, MIT

References

[R1]: Blake Nelson and Robert M. Kirby, "Ray-Tracing Polymorphic Multi-Domain Spectral/hp Elements for Isosurface Rendering", *IEEE Transactions on Visualization and Computer Graphics*, Vol. 12, Number 1, pages 114-125, 2006.

[R2]: S. Parker, M. Parker, Y. Livnat, P.P. Sloan, C.D. Hansen, and P. Shirley. "Interactive ray tracing for volume visualization". *IEEE Transactions on Visualization and Computer Graphics*, 5(3):238–250, July-September 1999.

[R3]: David H. Laidlaw, Robert M. Kirby, Cullen D. Jackson, J. Scott Davidson, Timothy S. Miller, Marco da Silva, William H. Warren, and Michael J. Tarr. “Comparing 2d vector field visualization methods: A user study”. *IEEE Transactions on Visualization and Computer Graphics*, 11(1):59–70, 2005.

[R4]: D. Weiskopf and G. Erlebacher. “Overview of flow visualization”. In C. D. Hansen and C. R. Johnson, editors, The Visualization Handbook. Elsevier, 2005.

[R5]: C. W. Gear. “Solving ordinary differential equations with discontinuities”. *ACM Transactions on Mathematical Software*, 10(1):23–44, 1984.

[R6]: J.H. Bramble and A.H. Schatz. “Higher order local accuracy by averaging in the finite element method”. *Mathematics of Computation*, 31:94–111, 1977.

[R7]: J.K. Ryan, C.-W. Shu, and H.L. Atkins. “Extension of a post-processing technique for the discontinuous Galerkin method for hyperbolic equations with application to an aeroacoustic problem”. *SIAM Journal on Scientific Computing*, 26:821–843, 2005.

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b. ADDRESS *(Include ZIP Code)* 75 S Central Campus Dr, Room 3750 Salt Lake City, UT 84112	d. AWARD DATE *(YYYYMMDD)* 20050630	b. ADDRESS *(Include ZIP Code)* 75 S Central Campus Dr, Room 3750 Salt Lake City, UT 84112	d. AWARD DATE *(YYYYMMDD)* 20050630	4. REPORTING PERIOD *(YYYYMMDD)* a. FROM 20050630 b. TO 20080731

SECTION I - SUBJECT INVENTIONS

5. "SUBJECT INVENTIONS" REQUIRED TO BE REPORTED BY CONTRACTOR/SUBCONTRACTOR *(If "None," so state)*

NAME(S) OF INVENTOR(S) *(Last, First, Middle Initial)* a.	TITLE OF INVENTION(S) b.	DISCLOSURE NUMBER, PATENT APPLICATION SERIAL NUMBER OR PATENT NUMBER c.	ELECTION TO FILE PATENT APPLICATIONS *(X)* d. (1) UNITED STATES (a) YES	(1) UNITED STATES (b) NO	(2) FOREIGN (a) YES	(2) FOREIGN (b) NO	CONFIRMATORY INSTRUMENT OR ASSIGNMENT FORWARDED TO CONTRACTING OFFICER *(X)* e. (a) YES	e. (b) NO
None	None	None						

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(1) (a) NAME OF INVENTOR *(Last, First, Middle Initial)*	(2) (a) NAME OF INVENTOR *(Last, First, Middle Initial)*	(1) TITLE OF INVENTION	(2) FOREIGN COUNTRIES OF PATENT APPLICATION
(b) NAME OF EMPLOYER	(b) NAME OF EMPLOYER		
(c) ADDRESS OF EMPLOYER *(Include ZIP Code)*	(c) ADDRESS OF EMPLOYER *(Include ZIP Code)*		

SECTION II - SUBCONTRACTS *(Containing a "Patent Rights" clause)*

6. SUBCONTRACTS AWARDED BY CONTRACTOR/SUBCONTRACTOR *(If "None," so state)*

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Robert Haimes	77 Massachusetts Ave Cambridge, MA 02139	2411099					20080731

SECTION III - CERTIFICATION

7. CERTIFICATION OF REPORT BY CONTRACTOR/SUBCONTRACTOR *(Not required if: (X as appropriate))* ☐ SMALL BUSINESS or ☒ NONPROFIT ORGANIZATION

I certify that the reporting party has procedures for prompt identification and timely disclosure of "Subject Inventions," that such procedures have been followed and that all "Subject Inventions" have been reported.

a. NAME OF AUTHORIZED CONTRACTOR/SUBCONTRACTOR OFFICIAL *(Last, First, Middle Initial)*	b. TITLE	c. SIGNATURE	d. DATE SIGNED
Mike Kirby	Associate Professor of Computer Science	[signature]	8/7/08